Indian Statistical Institute, Bangalore
B. Math. First Year
First Semester - Analysis I

Back paper Exam Date : Jan 05, 2015

This paper carries 41 marks. Max marks you can get is 30

1. For any 2×2 square matrix A let $|A| = detA$.
   Let$f_{11}, f_{12}, f_{21}, f_{22} : \mathbb{R} \to \mathbb{R}$ be differentiable functions.
   Let $A = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ Let the rows R_1, R_2 and columns C_1, C_2 be given by $R_1 = [f_{11}, f_{12}], R_2 = [f_{21}, f_{22}], C_1 = \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix}, C_2 = \begin{bmatrix} f_{12} \\ f_{22} \end{bmatrix}$.
   Let E_1, E_2, F_1, F_2 be the 2×2 square matrices given by $E_1 = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix}, E_2 = \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}, F_1 = [C_1', C_2], F_2 = [C_1, C_2']$
   (a) Find a relation between $|A|', |E_1|, |E_2|$
   (b) Find a relation for $|F_1|, |F_2|, |A|'$ [2]

   Discuss the summability of the following examples 2, 3, 4.

2. $\sum\limits_{n=1}^{\infty} p^n n^p \qquad p > 0$ [3]

3. $\sum\limits_{100}^{\infty} \frac{1}{n \log n (\log\log n)^p} \qquad p > 0$ [3]

4. $\sum\limits_{100}^{\infty} n^p (\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}})$ [3]

5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined by $x_1 = \frac{1}{2}$ and, for any $n \geq 1$,
$$x_{n+1} = \frac{x_n^2}{x_n^2 - x_n + 1}$$
   prove that $\sum_{n=1}^{\infty} x_n$ is convergent. [5]

6. (a) Let u_n be a sequence of complex numbers with $\sum |u_n| < \infty$. Show that $\sum\limits_1^{\infty} u_n^2$ exists. [3]
   (b) Give an example $a_1, a_2, ...$ a sequence of real numbers such that $\sum\limits_1^{\infty} a_n$ exists but $\sum a_n^2 = \infty$ and prove your claim. [2]

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying
$$f(x + y) = f(x) + f(y)$$
   for all x, y in $\mathbb{R}$. If f is continuous at x_0, show that f is continuous on the whole of $\mathbb{R}$. [3]

8. (a) Let $f : [0, 1] \to [0, 1]$ be continuous. By considering the function $g(x) = f(x) - x$ or otherwise show that there exists x_0 with $f(x_0) = x_0$ [1]

(b) Let f be as above and satisfying $f(f(y)) = f(y)$ for all y. Let $E_f = \{x : f(x) = x\}$. If E_f has at least two points then show that it must be an interval. [3]

9. Let $f : [0,1] \to \mathbb{R}$ be continuous in $[0,1]$ and differentiable in$(0,1)$ such that $f(0) = 0$ and $0 \leq f'(x) \leq 2f(x)$, for all $x \in (0,1)$. Prove that $f(x) = 0$ for all $x \in [0,1]$. [Hint: $g(x) = e^{-2x} f(x)$ may be useful.] [3]

10. Show that if f is continuous on $[0,\infty)$ and uniformly continuous on $[a,\infty)$ for some positive constant a, then f is uniformly continuous on $[0,\infty)$. [4]

11. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that there is no $x \in [0,1]$ such that $f(x) = f'(x) = 0$. Show that the set $Z := \{x \in [0,1] : f(x) = 0\}$ is finite. [3]

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(r + \frac{1}{n}) = f(r)$ for any rational number r and positive integer n. Prove that f is constant. [Hint: Is $f(r - \frac{1}{n}) = f(r)$ also for rational r and $n = 1,2,3...$] [3]